Applied Category Theory

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Mini-project
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Abstract

In this mini-project, we will be exploring the concept of Cauchy completion in category theory. We will be using the following definition given in the project assignment:

Definition 0.1: Basic definitions

A morphism $e : X \to X$ in a category $\mathcal{C}$ is called *idempotent* if $e \circ e = e$. A *splitting* of an idempotent morphism $e : X \to X$ is an object E together with maps $\iota : E \to X$ and $\pi : X \to E$ such that $\pi \circ \iota = \mathrm{id}_E$ and $\iota \circ \pi = e$. A category is called *Cauchy-complete* if every idempotent has a splitting.

We will begin by exploring some concrete examples of Cauchy-complete categories, specifically the category of matrices and stochastic matrices. We will then prove a more general result involving a characterisation of when a morphism e splits. Finally, we will explore some further constructions such as Cauchy-complete $\mathcal{V}$-categories and Karoubi envelopes, which will take us outside the material familiar to us from the course. The aim of the mini-project is to investigate how Cauchy-completeness appears in various different branches of category theory, how it can be interpreted in each case, and how seemingly unrelated definitions yield interesting and similar-looking results.

Question 1

Show that the category of (finite, real-entried) matrices is Cauchy-complete. Give a geometric interpretation of the splitting of idempotents.

First, recall the definition of the category of matrices:

Definition 1.1: Category of finite, real-entried matrices

The category **Mat** is defined as follows:

- $\text{Ob}(\textbf{Mat}) = \mathbb{N}$
- For all $n, m \in \text{Ob}(\textbf{Mat})$, the set of morphisms is defined as $\textbf{Mat}(n, m) = \mathbb{R}^{n \times m}$
- For all $n \in \text{Ob}(\textbf{Mat})$, the identity morphism is $\text{id}_n = I_n$, the $n \times n$ identity matrix
- Composition is given by matrix multiplication

It is a standard result that **Mat** is a category.

To prove that **Mat** is Cauchy-complete, it suffices to show that for any idempotent matrix $A \in \mathbb{R}^{n \times n}$, there exist matrices $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$ such that $BC = A$ and $CB = I_m$. We will give this construction explicitly.

First, we need to show that we can diagonalise A:

Lemma 1.2: Diagonalisability

Let A be an idempotent matrix. Then A is diagonalisable.

Proof:

Since A is idempotent, we know that $A^2 = A$ and therefore $A(A - I) = 0$. Thus, $p(x) = x(x - 1)$ is an annihilating polynomial for A. It is well known that a matrix is diagonalisable iff its minimal polynomial splits into distinct linear factors. Hence, any minimal polynomial must divide $p(x)$ and therefore must split into linear factors. Therefore A is diagonalisable. □

Based on Lemma 1.2, we know that $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D. The following lemma and corollary will help us determine the structure of D:

Lemma 1.3: Eigenvalues of A

Let A be an idempotent matrix. Then A has eigenvalues of $0, 1$ only.

Proof:

Suppose $Av = \lambda v$ for some vector $v \neq 0$. Then, since $A^2 = A$, we have that $\lambda v = Av = A^2 v = A(\lambda v) = \lambda(Av) = \lambda^2 v$. Thus $(\lambda^2 - \lambda)v = 0$, $\lambda(\lambda - 1) = 0$ and therefore $\lambda = 0$ or $\lambda = 1$. □

Corollary 1.4: Entries of the diagonal matrix

If $A = PDP^{-1}$ is diagonalisable, then the entries of D are the eigenvalues of A. Thus, they must all be 0 or 1. □

Now, suppose A has no eigenvalues of 0. The only such idempotent matrix is the identity matrix and thus the trivial solution of $A = B = C = I_n$ works. So, suppose A has at least one zero eigenvalue. We can now split the matrix D as follows:

Lemma 1.5: Splitting D

Let D be a matrix with ones and zeroes on the diagonal. Then D can be split into two rectangular matrices S, T such that $ST = D$ and $TS = I_m$ for some $m \leq n$.

Proof:

Let $m = \text{rank}(D)$, which is the number of ones on the diagonal, and $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ be the indices of the ones on the diagonal of D.

Define S to be the matrix that contains the $i_1, \ldots, i_m$'th columns of the identity matrix:

$$S = [e_{i_1}, \ldots, e_{i_r}] \in \mathbb{R}^{n \times m}$$

and then let $T = S^\top \in \mathbb{R}^{m \times n}$.

We know that each row of T is equal to each column of S, and there is exactly one 1 in each, with the rest of the values being 0. Furthermore, there are no two values i_k, i_l such that $k \neq l$ but $i_k = i_l$. Thus the product $TS = I_n$.

Now, $(ST)_{pq} = \sum_{k=1}^{m} S_{pk} T_{kq} = \sum_{k=1}^{m} S_{pk} S_{qk}$. Observe that $S_{pk} = 1$ if $p = i_k$, and 0 otherwise. Therefore

$$(ST)_{pq} = \begin{cases} 1, & p = q \in \{i_1, \ldots, i_m\} \\ 0, & \text{otherwise} \end{cases}$$

which is the exact definition of D. Hence $D = ST$ and $TS = I_m$. □

We are now able to show that A splits.

Theorem 1.6: Splitting of idempotent matrices

Suppose A is an idempotent matrix. Then there exist matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ such that $BC = A$ and $CB = I_m$.

Proof:

Recall from Lemma 1.2 and Lemma 1.5 that we have the following equations:

$$A = PDP^{-1}$$
$$D = ST$$
$$TS = I_m$$

We now let

$$B = PS$$
$$C = TP^{-1}$$

Observe that $BC = PSTP^{-1} = PDP^{-1} = A$, and $CB = TP^{-1}PS = TI_nS = TS = I_m$. Hence A, B, and C satisfy our criteria, and in the category-theoretical setting these correspond to e, π, and ι, respectively. □

We can interpret this splitting geometrically as follows. A graphical example is shown in Figure 1.

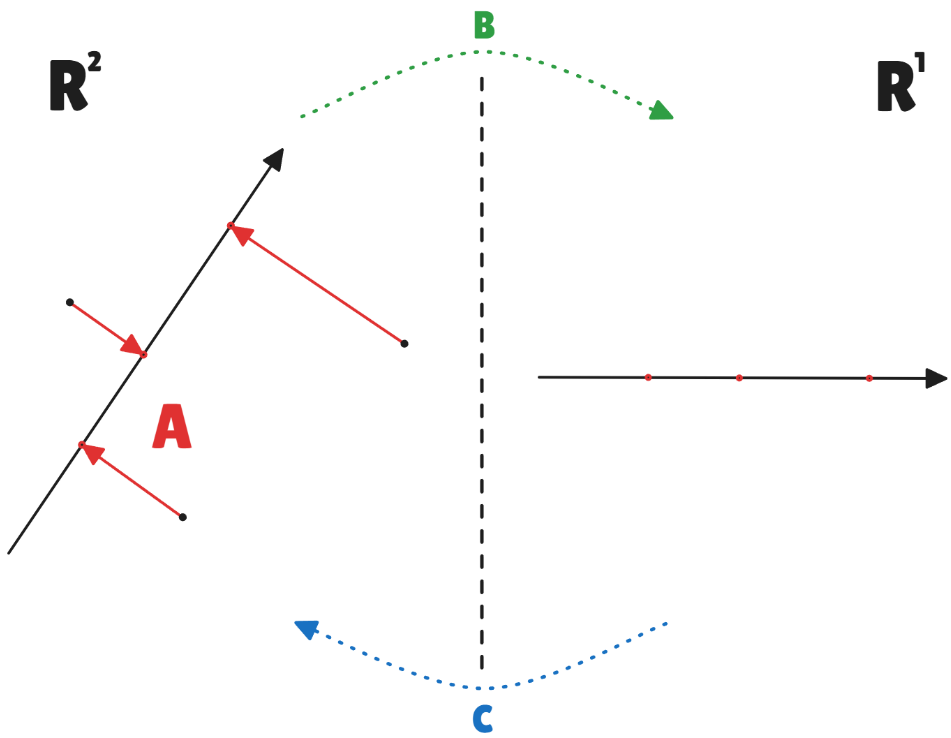


Figure 1: A graphical interpretation of the matrices A, B, C

Any idempotent matrix is a projection of vectors onto a hyperplane: that is, all points in the vector space are projected onto an m-dimensional plane in $\mathbb{R}^n$. Our matrix B is a projection mapping from the space $\mathbb{R}^n$ to the space $\mathbb{R}^m$. Intuitively, in the diagram, it turns the m-dimensional hyperplane represented by the black arrow into a regular m-dimensional vector space on the right, while projecting all points outside this hyperplane in the same way as the matrix A. Conversely, the matrix C is a suitable embedding of the space $\mathbb{R}^m$ into our original space $\mathbb{R}^n$ preserving the original orientation of the hyperplane.

The product BC gives us a matrix that, when applied to a vector in $\mathbb{R}^n$, first projects it into $\mathbb{R}^m$ and then embeds it back into $\mathbb{R}^n$ in a suitable orientation. This is the same as projecting the vector onto the plane directly, and hence $BC = A$.

The product CB gives us a matrix that, when applied to a vector in $\mathbb{R}^m$, first embeds it into $\mathbb{R}^n$ and then projects it back to $\mathbb{R}^m$. Obviously, this does nothing, since an embedding and a projection are inverse operations in this direction. Hence $CB = I_m$.

Question 2

Is the subcategory of stochastic matrices Cauchy-complete as well? If you can, give an interpretation in terms of Markov chains. (If you are not familiar with Markov chains, you can reason purely in terms of matrices, as in the previous point.)

The subcategory of stochastic matrices is Cauchy-complete. To prove this, we will also give an explicit construction for any given idempotent stochastic matrix and prove that it works correctly.

First, let us prove some auxiliary lemmas:

Lemma 2.1: Rows remain unchanged

Let $A \in \mathbb{R}^{n\times n}$ be an idempotent matrix, and v_k be one of its rows. Then $v_k A = v_k$.

Proof:

Since v_k is one of the rows of A, it belongs to its row space. One characterisation of row space is that $v_k = uA$ for some $u \in \mathbb{R}^n$. Then $v_k A = (uA)A = uA^2 = uA = v_k$. □

Now, let $\{v_1, \ldots, v_m\}$ be the linearly independent rows of A. We will let C be the matrix whose rows are $v_1, \ldots, v_m$ as follows:

$$C = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

Note that, since each row of C comes from A, each row will sum to 1 and therefore C is stochastic.

Now, let us write each row of A as a linear combination of our linearly independent rows. Let us label the rows of A as a_i for all $i \in \{1, \ldots, n\}$. Then there exist coefficients $w_{i,k} \in \mathbb{R}$ such that

$$a_i = \sum_{k=1}^{m} w_{i,k} v_k \tag{1}$$

Let us show a nice property of these coefficients.

Lemma 2.2: The weights sum to 1

Consider the weights $w_{i,k} \in \mathbb{R}$ defined above. For any $i \in \{1, \ldots, n\}$, the sum

$$\sum_{k=1}^{m} w_{i,k} = 1$$

Proof:

Recall that due to A being stochastic, we have that

$$\sum_{j=1}^{n} A_{ij} = 1$$

Hence,

$$1 = \sum_{j=1}^{n} A_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{m} w_{i,k}(v_k)_j = \sum_{k=1}^{m} \sum_{j=1}^{n} w_{i,k}(v_k)_j =$$
$$= \sum_{k=1}^{m} w_{i,k} \left(\sum_{j=1}^{n} (v_k)_j \right) = \sum_{k=1}^{m} w_{i,k}$$

With the last step holding since the rows of A always sum to 1. □

We now define the matrix B to simply be the matrix of these weights:

$$B_{ij} = w_{i,j}$$

Due to Lemma 2.2, B is a stochastic matrix.

We are now ready to prove the result that $A = BC$ and $CB = I_m$.

Theorem 2.3: The first condition holds

Let A be a stochastic idempotent matrix. Then, with the B, C defined above, $A = BC$.

Proof:

We prove this by computing the product directly:

$$\begin{aligned}
(BC)_{ij} &= \sum_{k=1}^{m} B_{ik} C_{kj} = && \text{(definition of B and C)} \\
&= \sum_{k=1}^{m} w_{i,k}(v_k)_j = && \text{(scalar multiplication)} \\
&= \sum_{k=1}^{m} (w_{i,k} v_k)_j = && \text{(equation 1)} \\
&= A_{ij}
\end{aligned}$$

□

Theorem 2.4: The second condition holds

Let A be a stochastic idempotent matrix. Then, with the B, C defined above, $CB = I_m$.

Proof:

Begin by observing that

$$(CB)_{ij} = \sum_{k=1}^{n} C_{ij} B_{kj} = \sum_{k=1}^{n} (v_i)_k w_{k,j}$$

Since we defined $a_k = \sum_{l=1}^{m} w_{k,l} v_l$, we have that

$$v_i A = \sum_{k=1}^{n} (v_i)_k a_k = \sum_{k=1}^{n} (v_i)_k \sum_{l=1}^{m} w_{k,l} v_l = \sum_{l=1}^{m} \left(\sum_{k=1}^{n} (v_i)_k w_{k,l} \right) v_l$$

In addition, recall that by Lemma 2.1 we have $v_i A = v_i$. Also, the vector v_i can be rewritten as $v_i = \sum_{l=1}^{m} \delta_{il} v_l$, where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. Thus,

$$\sum_{l=1}^{m} \delta_{il} v_l = \sum_{l=1}^{m} \left(\sum_{k=1}^{n} (v_i)_k w_{k,l} \right) v_l$$

Since the v_l's were defined to be linearly independent, the coefficients must be equal. Hence

$$\sum_{k=1}^{n} (v_i)_k w_{k,l} = \delta_{il}$$
$$\sum_{k=1}^{n} (v_i)_k w_{k,j} = \delta_{ij}$$

Therefore $(CB)_{ij} = \delta_{ij}$, which is exactly the definition of the $m \times m$ identity matrix. $\square$

In a Markov chain, an idempotent matrix A represents a scenario where the system reaches a stable state, i.e. where the probabilities do not change after taking a single step.

The matrix C projects any distribution onto a smaller state space of size m. While A maps any distribution into a fixed distribution, the matrix C extracts the essential structure of the distribution into a minimal basis.

Conversely, B embeds the distribution from the minimal basis into the original basis in $\mathbb{R}^n$.

We may also reason purely in terms of matrices, but the logic is exactly the same as in part 1 of this mini-project.

Question 3

Given an idempotent morphism $e : X \to X$, define the *presheaf of left-invariant morphisms* $\mathrm{Inv}_e : \mathcal{C}^{op} \to \mathbf{Set}$ as follows,

$$\mathrm{Inv}_e(Y) = \{p : Y \to X \mid e \circ p = p\}$$

with its action on morphisms given by precomposition. Show that e splits if and only if Inv_e is representable. Give an interpretation in terms of "virtual objects".

Recall that a presheaf is representable if it is naturally isomorphic to $\mathrm{Hom}(-, B)$ for some B, defined as follows:

Definition 3.1: The contravariant hom-functor $\mathrm{Hom}(-, B)$

The contravariant functor $\mathrm{Hom}(-, B) : \mathcal{C}^{op} \to \mathbf{Set}$ is defined as follows:

- $\mathrm{Hom}(-, B)$ maps each object X in $\mathcal{C}$ to the set of its morphisms, $\mathcal{C}(X, B) = \mathrm{Hom}(X, B)$
- $\mathrm{Hom}(-, B)$ maps each morphism $f \in \mathcal{C}^{op}(Y, X)$ to the map $\mathrm{Hom}(f, B) : \mathrm{Hom}(Y, B) \to \mathrm{Hom}(X, B)$ defined by $g \mapsto g \circ f$ in $\mathcal{C}$.

We will prove each direction of the desired statement independently.

($\Rightarrow$) Suppose e splits.

So there exists an object E with morphisms $\pi : X \to E$ and $\iota : E \to X$ such that $e = \iota \circ \pi$ and $\pi \circ \iota = \mathrm{id}_E$. Define the presheaf Inv_e by $\mathrm{Inv}_e(Y) = \{p : Y \to X \mid e \circ p = p\} = \{p : Y \to X \mid \iota \circ \pi \circ p = p\}$. Then, define the natural transformation $\alpha : \mathrm{Hom}(-, E) \Rightarrow \mathrm{Inv}_e$ componentwise by stating that for each object $A \in \mathcal{C}$,

$$\alpha_A(h) = \iota \circ h \tag{2}$$

We now need to show three things: that α is well-defined, that it is natural, and that each morphism α_A is an isomorphism or a bijection.

Lemma 3.2: α is well-defined

We claim that $\iota \circ h \in \mathrm{Inv}_e(A)$; that is,

$$e \circ (\iota \circ h) = \iota \circ h$$

Proof:

This can be shown directly by computing the composition:

$$e \circ (\iota \circ h) = (\iota \circ \pi) \circ (\iota \circ h) = \iota \circ (\pi \circ \iota) \circ h = \iota \circ \mathrm{id}_E \circ h = \iota \circ h \quad \square$$

Now, let us move onto naturality. Recall that for a natural transformation, the following diagram must commute for any $f \in \mathcal{C}(A, B)$:

$$\begin{array}{ccc} \mathrm{Hom}(A, E) & \xrightarrow{\alpha_A} & \mathrm{Inv}_e(A) \\ \downarrow{\circ f} & & \downarrow{\circ f} \\ \mathrm{Hom}(B, E) & \xrightarrow{\alpha_B} & \mathrm{Inv}_e(B) \end{array}$$

Figure 2: Naturality square

Lemma 3.3: α is natural

Consider our natural transformation α defined above. For any morphism $f \in \mathcal{C}(A, B)$, the diagram in Figure 2 commutes.

Proof:

Take any $g : A \to E$. Then, by going down and right we get:

$$\alpha_B(h \circ f) = \iota \circ (h \circ f) = (\iota \circ h) \circ f$$

Conversely, going right and down we get:

$$\alpha_A(h) \circ f = (\iota \circ h) \circ f$$

Since these are equal, the square commutes and α is natural. $\square$

Finally, we must prove the isomorphism property.

Theorem 3.4: α is a natural isomorphism

For all objects $A \in \mathcal{C}$, the function α_A in **Set** is injective and surjective.

Proof:

Suppose for some morphisms $f, g : A \to E$, $\alpha_A(f) = \alpha_A(g)$. Then

$$\iota \circ f = \iota \circ g$$

$$\pi \circ \iota \circ f = \pi \circ \iota \circ g$$

$$\mathrm{id}_E \circ f = \mathrm{id}_E \circ g$$

$$f = g$$

So α_A is injective. $\square$

Now, let $f \in \mathrm{Inv}_e(A), f : A \to X$ such that

$$e \circ f = f$$

We now let $g = \pi \circ f : A \to E$, which is the "preimage" of f. To see this, we compute directly:

$$\alpha_A(g) = \iota \circ (\pi \circ f) = e \circ f = f$$

Hence each function α_A is both injective and surjective, and thus bijective and an isomorphism.

$\square$

Thus we have proved that there exists a natural isomorphism between Inv_e and $\mathrm{Hom}(-, E)$, which means that Inv_e is representable. $\square$

We can now do the converse direction.

(⇐) Suppose Inv_e is representable.

So there exists a natural isomorphism $\alpha : \mathrm{Hom}(-, E) \Rightarrow \mathrm{Inv}_e$ for some object $E \in \mathcal{C}$. Observe that, specifically, there exists a function $\alpha_E : \mathrm{Hom}(E, E) \to \mathrm{Inv}_e(E) = \{f : E \to X \mid e \circ f = f\}$. We now take the identity morphism $\mathrm{id}_E \in \mathcal{C}(E, E)$ and define $\iota = \alpha_E(\mathrm{id}_E) : E \to X$. We also know that $\iota \in \mathrm{Inv}_e(E)$, thus $e \circ \iota = \iota$.

To define π, we need to take the inverse natural transformation of α. Since each α_A is a bijection, we can simply define α^{-1} by taking $\alpha_A^{-1} = (\alpha_A)^{-1}$. Now, since e is idempotent, $e \circ e = e$ and therefore $e \in \mathrm{Inv}_e(X)$. Therefore we can define $\pi = \alpha_X^{-1}(e) : X \to E$, which also means that $\alpha_X(\pi) = e$.

We now need to show that the required splitting properties hold.

Theorem 3.5: The first splitting property holds

We claim that, with the above construction, $\iota \circ \pi = e$.

Proof:

Consider the naturality square given in Figure 3, which must commute as α is a natural transformation. We know that the morphism $\mathrm{id}_E \in \mathrm{Hom}(E, E)$, so let us apply both paths of the naturality square to it:

$$\alpha_E(\mathrm{id}_E) \circ \pi = \alpha_X(\mathrm{id}_E \circ \pi)$$
$$\iota \circ \pi = \alpha_X(\pi) = e$$

Which was what we wanted. □

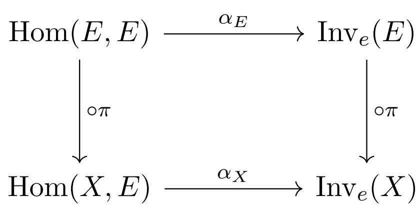


Figure 3: Naturality square for proof of Theorem 3.5

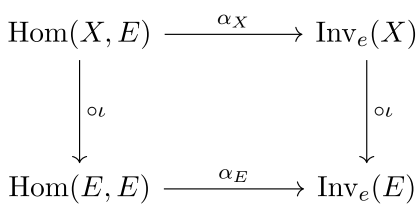


Figure 4: Naturality square for the proof of Theorem 3.6

Theorem 3.6: The second splitting property holds

We claim that, with the above construction, $\pi \circ \iota = \mathrm{id}_E$.

Proof:

This time, consider the naturality square given in Figure 4. Since $\pi \in \mathrm{Hom}(X, E)$, we can trace this diagram in a similar way:

$$\begin{aligned} \alpha_X(\pi) \circ \iota &= \alpha_E(\pi \circ \iota) \\ e \circ \iota &= \alpha_E(\pi \circ \iota) \\ \iota &= \alpha_E(\pi \circ \iota) \\ \alpha_E(\mathrm{id}_E) &= \alpha_E(\pi \circ \iota) \end{aligned}$$

Since we know that α_E is bijective, we can take its inverse to conclude that

$$\pi \circ \iota = \mathrm{id}_E$$

as required. □

Hence, we have proven that both splitting properties hold, and therefore the morphism e splits given that Inv_e is a representable functor. □

Interestingly, presheaves can be interpreted as an extension of a category using a virtual object. We can see this formalised in the following definition [12, Definition 2.1]:

Definition 3.7: Category with a virtual object

Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. The category $\mathcal{C}^{+F}$ has:

- As objects, the ones of $\mathcal{C}$ plus an extra object E
- As morphisms between objects coming from $\mathcal{C}$, the ones of $\mathcal{C}$
- As morphisms $E \to A$, where A is an object of $\mathcal{C}$, the elements of FA called virtual arrows
- The identity of E as the unique morphism with codomain E.

The composition of morphisms of $\mathcal{C}$ is as in $\mathcal{C}$, and between morphisms $E \to A$ and $A \to B$ is as specified by functoriality of F.

In this definition, any arrows in $\mathcal{C}$ are called real arrows, while any morphisms out of E are called virtual arrows. In the case of a presheaf, this definition is reversed, and arrows point into the extra object E instead of out of it [12, Definition 2.5]:

Definition 3.8: Category with a virtual object coming from a presheaf

Let $\mathcal{C}$ be a category, and let $P : \mathcal{C}^{op} \to \mathbf{Set}$ be a presheaf. The category $\mathcal{C}_{+P}$ has:

- As objects, the ones of $\mathcal{C}$ plus an extra object E
- As morphisms between objects coming from $\mathcal{C}$, the ones of $\mathcal{C}$
- As morphisms $A \to E$, where A is an object of $\mathcal{C}$, the elements of PA called virtual arrows
- The identity of E as the unique morphism with domain E.

The composition of morphisms of $\mathcal{C}$ is as in $\mathcal{C}$, and between morphisms $A \to B$ and $B \to E$ is as specified by functoriality of P.

With this definition, the paper then concludes the following: "A functor is representable if an only if its virtual object and arrows correspond to some real objects and arrows inside our category". More formally, the paper concludes the following [12, Theorem 2.7, Corollary 2.8]:

Theorem 3.9: Representable functors

A functor $F : \mathcal{C} \to \mathbf{Set}$ is represented by an object R if and only if the inclusion functor $I : \mathcal{C} \hookrightarrow \mathcal{C}^{+F}$ admits a retraction functor $\Pi : \mathcal{C}^{+F} \to \mathcal{C}$ mapping $\mathcal{C} \subseteq \mathcal{C}^{+F}$ to itself and E to R, and satisfying any of the following equivalent conditions:

(i) Π is fully faithful, meaning that for every A of $\mathcal{C}$, Π maps each virtual arrow $E \to A$ bijectively to a real arrow $R \to A$

(ii) Π is left-adjoint to the inclusion I

Corollary 3.10: Representable presheaves

Dually, a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ is representable if and only if the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}_{+P}$ admits a retraction which is either fully faithful, or equivalently, right-adjoint

We observe that in this mini-project we have proven a special case of this result, namely that while our presheaf Inv_e can be represented by a virtual object in general, it can be represented by an actual object E with a splitting morphism e if and only if it is itself representable.

Question 4

> By doing a little research, find out what a Cauchy-complete $\mathcal{V}$-category is, and explain how the notion relates to completeness of metric spaces.

To define a Cauchy-complete $\mathcal{V}$-category, we will first need to recall some definitions. The first is that of a cartesian closed category and that of an enriched category, which we will take from [3, Definition 6.2.1] and [7]. We treat the definition of a monoidal category as familiar from the course.

Definition 4.1: Enriched category

Let $\mathcal{V}$ be a monoidal category with

- Tensor product $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
- Tensor unit $I \in \mathcal{V}$
- Associator $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$
- Left unitor $l_a : I \otimes a \to a$
- Right unitor $r_a : a \otimes I \to a$

Which satisfies the usual axioms of a monoidal category. A $\mathcal{V}$-category $\mathcal{C}$ is then given by:

- A set of objects
- For each pair of objects $a, b \in \mathcal{C}$, a hom-object $\mathcal{C}(a, b) \in \mathcal{V}$
- For each $a \in \mathcal{C}$, an identity element which is a morphism $j_a : I \to \mathcal{C}(a, a)$
- For each $a, b, c \in \mathcal{C}$, a morphism $\circ : \mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \to \mathcal{C}(a, c)$ called the composition morphism
- Satisfying the associativity and unit conditions shown in Figure 5 and Figure 6, respectively.

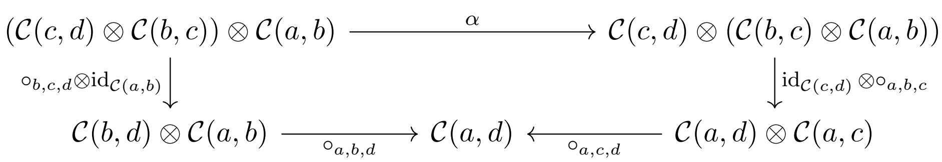


Figure 5: Associativity condition

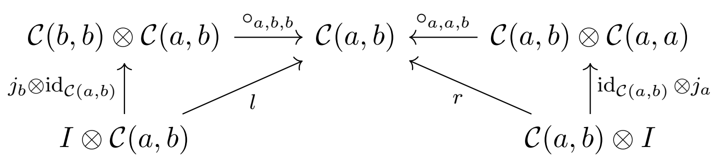


Figure 6: Unit condition

With this, we can define enriched functors as given in [3, Definition 6.2.3] and [8]:

Definition 4.2: Enriched functor

Given two categories $\mathcal{A}$ and $\mathcal{B}$ enriched in a monoidal category $\mathcal{V}$, an enriched functor $F : \mathcal{A} \to \mathcal{B}$ consists of:

- A function $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ between the underlying collections of objects
- A $(\mathcal{A}_0 \times \mathcal{A}_0)$-indexed collection of morphisms of $\mathcal{V}$

  $$F_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(F_0x, F_0y)$$

  where $\mathcal{A}(x, y)$ denotes the hom-object in $\mathcal{V}$
- Such that the diagrams in Figure 7 and Figure 8 commute.

$$\begin{array}{ccc} \mathcal{A}(b,c) \otimes \mathcal{A}(a,b) & \xrightarrow{\circ_{a,b,c}} & \mathcal{A}(a,c) \\ {\scriptstyle F_{b,c} \otimes F_{a,b}} \downarrow & & \downarrow {\scriptstyle F_{a,c}} \\ \mathcal{B}(F_0(b), F_0(c)) \otimes \mathcal{B}(F_0(a), F_0(b)) & \xrightarrow[\circ_{F_0(a),F_0(b),F_0(c)}]{} & \mathcal{B}(F_0(a), F_0(c)) \end{array}$$

Figure 7: Associativity condition for enriched functors

$$\begin{array}{ccc} & I & \\ {\scriptstyle j_a} \swarrow & & \searrow {\scriptstyle j_{F_0(a)}} \\ \mathcal{A}(a,a) & \xrightarrow[F_{a,a}]{} & \mathcal{B}(F_0(a), F_0(a)) \end{array}$$

Figure 8: Unit condition for enriched functors

Now, we move onto define what a module over an enriched category is. This will be taken from [3, Definition 6.2.19] and [11], and is a generalisation of other similar constructions such as those seen in [1].

Definition 4.3: Module over an enriched category

An enriched module (also known as a $\mathcal{V}$-module, a distributor, a $\mathcal{V}$-profunctor, or an $\mathcal{A}$-$\mathcal{B}$-bimodule) from a category $\mathcal{A}$ to a category $\mathcal{B}$ is a $\mathcal{V}$-enriched functor

$$\mathcal{A}^{op} \otimes \mathcal{B} \to \mathcal{V}$$

We also use the notation $\mathcal{A} \nrightarrow \mathcal{B}$ to denote such a module.

Finally, we can define what being *Cauchy* means for modules and enriched categories. These definitions are taken from the paper *Cauchy Completeness and Causal Spaces* [6, Appendix A].

Definition 4.4: Cauchy $\mathcal{V}$-module

A $\mathcal{V}$-module $M : \mathcal{A} \to \mathcal{B}$ is called Cauchy if it has a right adjoint $\mathcal{V}$-functor.

Definition 4.5: Cauchy complete $\mathcal{V}$-category

A $\mathcal{V}$-category $\mathcal{C}$ is Cauchy complete if all Cauchy modules into $\mathcal{C}$ are representable.

Note that we do not explicitly define what adjointness means for $\mathcal{V}$-modules here. The definition is similar to that of adjointness in regular category theory, but requires quite a lot of technical detail when defining $\mathcal{V}$-natural transformations. Instead of defining it here, we will later see a useful characterisation of adjointness of $\mathcal{V}$-modules for the specific case of an enriched category representing a metric space. We will now move on to show the connection between enriched categories and metric spaces. First, we will establish an observation from a 1973 paper by Lawvere [5] to show that, in fact, metric spaces are just an example of an enriched category. The first step is to define the monoidal category $\bar{R}_+$:

Definition 4.6: The monoidal category $\bar{R}_+$

Define the category $\bar{R}_+$ as follows:

- $\mathrm{Ob}(\bar{R}_+) = [0, \infty]$, i.e. the nonnegative real numbers together with the infinity object
- Morphisms are given by $\geq$ and compose in the usual way
- The tensor product is given by addition, $\otimes = +$
- The tensor unit is $I = 0$
- The associator $\alpha_{a,b,c}$, left unitor l_a, and right unitor r_a are all the respective identity morphisms

It is easy to see that this defines a monoidal category: $\bar{R}_+$ is a category as it is a poset with an additional point at infinity, and the monoidal structure holds due to associativity of addition of natural numbers. New, we will show that a metric space is just a category enriched in $\bar{R}_+$ [2].

Proposition 4.7: Metric spaces as enriched categories

Let (E, d) be a metric space. Then, define an enriched $\bar{R}_+$-category $\mathcal{E}$ using Definition 4.1 as follows: Let E be the set of objects and for all $a, b \in E$, let $\mathcal{E}(a, b) = d(a, b) \in \bar{R}_+$ be the hom-object. This is a well-defined enriched category.

Proof:

First, observe that Definition 4.1 requires us to define an identity element and composition morphism. For the former, we notice that because (E, d) is a metric space, $\mathcal{E}(a, a) = d(a, a) = 0$ for all a. Since also the tensor unit $I = 0$, the identity element j_a is just the unique morphism $\geq$ that witnesses $0 \geq 0$ for all a. Similarly, the composition morphism $\circ : \mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \to \mathcal{C}(a, c)$

is just the unique morphism $\geq$ that witnesses $d(a,b)+d(b,c)\geq d(a,c)$, which must exist by the triangle inequality for metric spaces.

Now, let us show that the associativity and unit conditions are satisfied. Starting with Figure 5, we will first trace the right-down path:

$$\circ_{a,c,d}\left((\mathrm{id}_{\mathcal{E}(c,d)}\otimes\circ_{a,b,c})(\alpha((\mathcal{E}(c,d)\otimes\mathcal{E}(b,c))\otimes\mathcal{E}(a,b)))\right)=$$
$$=\circ_{a,c,d}((\mathrm{id}_{d(c,d)}\otimes\circ_{a,b,c})(d(c,d)+d(b,c)+d(a,b)))$$

This expression asks for a witness of the following chain of inequalities:

$$d(c,d)+d(b,c)+d(a,b)\geq d(c,d)+d(a,c)\geq d(a,d)$$

Which is true by the triangle inequality. Then, we trace the down-right path:

$$\circ_{a,b,d}\left((\circ_{b,c,d}\otimes\mathrm{id}_{\mathcal{E}(a,b)})((\mathcal{E}(c,d)\otimes\mathcal{E}(b,c))\otimes\mathcal{E}(a,b))\right)=$$
$$=\circ_{a,b,d}((\circ_{b,c,d}\otimes\mathrm{id}_{d(a,b)})(d(c,d)+d(b,c)+d(a,b))$$

Which correspondingly asks for a witness of the following inequality chain:

$$d(c,d)+d(b,c)+d(a,b)\geq d(a,b)+d(b,d)\geq d(a,d)$$

Which is also true, but arrives to the conclusion by applying the triangle inequality in a different order. Since both paths witness the fact that

$$\circ_{a,b,d}((\circ_{b,c,d}\otimes\mathrm{id}_{\mathcal{E}(a,b)})((\mathcal{E}(c,d)\otimes\mathcal{E}(b,c))\otimes\mathcal{E}(a,b))))\geq\mathcal{E}(a,d)$$

and the morphism $\geq$ is unique, we conclude that this condition holds.

We now do a similar procedure for the unit condition shown in Figure 6. Starting at the bottom-left and going diagonally, we get the trivial result that

$$l(I\otimes\mathcal{E}(a,b))=l(0+d(a,b))=l(d(a,b))$$

Which is just a witness of the fact that $d(a,b)\geq d(a,b)$. Going via the top-left, we get the slightly less trivial result that

$$\circ_{a,b,b}\left((j_b\otimes\mathrm{id}_{\mathcal{E}(a,b)})(I\otimes\mathcal{E}(a,b))\right)=$$
$$=\circ_{a,b,b}((j_b\otimes\mathrm{id}_{d(a,b)})(d(a,b)))$$

Which asks for a witness of the following inequality chain:

$$d(a,b) \geq d(a,b) + d(b,b) \geq d(a,b)$$

Which is true due to $d(b,b) = 0$ and a subsequent application of the triangle inequality. Again, due to the uniqueness of $\geq$ we can conclude that these two paths are equal.

The case for the bottom-right object $\mathcal{E}(a,b) \otimes I$ is identical: it asks for a witness of the trivial chain $d(a,b) \geq d(a,b)$ and $d(a,b) \geq d(a,b) + d(a,a) \geq d(a,b)$, which holds for the exact same reason as above. □

Now that we have defined our enriched category, we may also define the enriched functors that we want to study.

Proposition 4.8: Contracting mappings are enriched functors

Let (E,d) and (E',d') be two metric spaces. Recall that a contracting mapping $F : (E,d) \to (E',d')$ is a function that satisfies the following equation for all $x,y \in E$ and some $0 \leq k < 1$:

$$kd(x,y) \geq d'(F(x),F(y))$$

Every such mapping F is an enriched functor as given in Definition 4.2.

Proof:

Looking at the definition, we let the map on the underlying set of objects $F_0 = F$. Note that due to the contraction condition, we know that $d(x,y) \geq kd(x,y) \geq d'(F(x),F(y))$ and thus the morphism $F_{x,y}$ is just the unique morphism $\geq$ witnessing this inequality.

We must now show that the associativity and unit conditions hold once again. Beginning with Figure 7, observe that the right-then-down path is given by

$$F_{a,c}(\circ_{a,b,c}(\mathcal{A}(b,c) \otimes \mathcal{A}(a,b))) =$$
$$= F_{a,c}(\circ_{a,b,c}(d(b,c) + d(a,b)))$$

This asks for a witness $\geq$ of the following inequality chain:

$$d(b,c) + d(a,b) \geq d(a,c) \geq d'(F(a),F(c))$$

Which is true by the properties outlined above. The down-then-right path is given by

$$\circ_{F_0(a),F_0(b),F_0(c)}\left((F_{b,c} \otimes F_{a,b})(\mathcal{A}(b,c) \otimes \mathcal{A}(a,b))\right) =$$
$$= \circ_{F_0(a),F_0(b),F_0(c)}((F_{b,c} \otimes F_{a,b})(d(b,c) + d(a,b)))$$

which, in turn, asks for a witness $\geq$ of the inequality chain

$$d(b,c) + d(a,b) \geq d'(F(b), F(c)) + d'(F(a), F(b)) \geq d'(F(a), F(c))$$

Which still holds by properties of metric spaces. Again, since the morphism $\geq$ is unique, we conclude that this square commutes.

Now, looking at Figure 8, we repeat the same process starting with the down-right direction:

$$j_{F_0(a)}(I) = j_{F_0(a)}(0)$$

Which asserts the inequality

$$0 \geq d'(F(a), F(a))$$

which is true since $d'(x,x) = 0$ for all x. The down-left direction is similar:

$$F_{a,a}(j_a(I)) = F_{a,a}(j_a(0))$$

this asserts the following chain:

$$0 \geq d(a,a) \geq d'(F(a), F(a))$$

which again holds trivially. For the same reason as before, the triangle commutes, and we are done. □

We now define distributors (or modules, as was the terminology used in Definition 4.3) in the context of this enriched category. Consider the enriched category $\mathcal{I}$, which is formed by taking the singleton metric space as an enriched category. Then, given a category $\mathcal{E}$ from a metric space (E,d) we will look at the module $F : \mathcal{I} \nrightarrow \mathcal{E}$. By evaluating Definition 4.3, this ends up simply being the mapping $F : E \to \bar{R}_+$ which satisfies the following condition for all $x, y \in E$:

$$d(x,y) \geq \max\{F(y) - F(x), 0\}$$

Since d is symmetric, we can notice that the function must also satisfy the following property:

$$d(x, y) \geq |F(y) - F(x)| \tag{3}$$

We now claim without explicit proof a lemma about a useful fact related to adjoint distributors which are defined in the above fashion.

Lemma 4.9: Properties of adjoint distributors

F and G are distributors $F : \mathcal{I} \nrightarrow \mathcal{E}$ and $G : \mathcal{E} \nrightarrow \mathcal{I}$ such that G is right adjoint to F if and only if $F, G : E \to \bar{R}_+$ satisfying the following two properties:

$$\inf_{x \in X} \{F(x) + G(x) \mid x \in X\} = 0 \tag{4}$$

$$\forall x, y \in X, \quad F(x) + G(y) \geq d(x, y) \tag{5}$$

Proof:

The proof is given in [2]. A full proof relies on defining the composite of two modules, as well as enriched natural transformations to properly define adjointness as is done in [3, Definition 6.2.4, Proposition 6.2.11], both of which are outside of the scope of this mini-project. □

We are now ready to prove the main result of this section. This is a theorem in [2] that claims that the Cauchy completion of a metric space in the classical sense corresponds bijectively to all of the distributors defined above.

Theorem 4.10: Elements in a complete metric space and distributors

Let (E, d) be a metric space and $(\bar{E}, \bar{d})$ its Cauchy completion; that is, the metric space obtained from E by adding in the limits of all Cauchy sequences. The elements of $\bar{E}$ correspond bijectively with the pairs of adjoint distributors given in Lemma 4.9.

Proof:

(Injectivity)

Let $a \in \bar{E}$. We define a function $F_a : E \to \bar{R}_+$ with

$$F_a(x) = \bar{d}(a, x)$$

By equation 3, observe that for all $x, y \in E$

$$|F_a(x) - F_a(y)| \leq d(x, y) \leq F_a(x) + F_a(x)$$

Since $\bar{E}$ is complete, there exists a Cauchy sequence (a_n) in E converging to a. Therefore we have the following equation

$$\inf_{n\in\mathbb{N}}(F_a(a_n) + F_a(a_n)) = 0$$

These two results make the conditions in equations 4 and 5 hold if we let $F_a = G_a$. Thus we know that $F_a : \mathcal{I} \twoheadrightarrow \mathcal{E}$ and $G_a : \mathcal{E} \twoheadrightarrow \mathcal{I}$ with $F_a \dashv G_a$ adjoint. This correspondence is injective since $d(a,b) \neq 0$ for $a \neq b$, and thus when $F_a(a_n) \to 0$, the sequence $F_b(a_n) \to d(a,b) \neq 0$.

(Surjectivity)

Let $F \dashv G$ be adjoint distributors. For all $n \in \mathbb{N}$, choose an $a_n \in E$ such that the following equation holds, which is possible due to Lemma 4.9:

$$F(a_n) + G(a_n) < \frac{1}{n}$$

Now, consider any $k, l \geq n$:

$$d(a_k, a_l) \leq F(a_k) + G(a_l) \leq F(a_k) + G(a_k) + F(a_l) + G(a_l) \leq \frac{1}{k} + \frac{1}{l} \leq \frac{2}{n}$$

Thus the sequence (a_n) is Cauchy in E, and hence its limit $(a_n) \to a \in \bar{E}$. Then, for all $x \in E$

$$|F(x) - F(a_n)| \leq d(x, a_n) \leq F(x) + G(a_n)$$

Computing the limit of all the terms gives us

$$\lim_{n\to\infty} |F(x) - F(a_n)| \leq \lim_{n\to\infty} d(x, a_n) \leq \lim_{n\to\infty} F(x) + G(a_n)$$
$$F(x) \leq \lim_{n\to\infty} d(x, a_n) \leq F(x)$$

and hence by the squeeze theorem

$$F(x) = \lim_{n\to\infty} d(x, a_n) = \bar{d}(x, a) = F_a(x)$$

We must finally prove that the distance between the distributors, defined as the supremum of positive distances between points, is the same as $\bar{d}$. Observe that

$$\sup_{x\in E}\{\max\{F_b(x) - F_a(x), 0\}\} = \sup_{x\in E}\{\max\{\bar{d}(b,x) - \bar{d}(a,x), 0\}\} \leq \bar{d}(b,a)$$

If $a = b$ then both distances are zero. However, if $a \neq b$, by properties of metric spaces we get $\bar{d}(b, a_n) - \bar{d}(a_n) \to \bar{d}(b,a)$. So the distance between the two distributors is equal to $\bar{d}(b,a)$. $\square$

We may also state one final interesting relationship between distributors and completeness of metric spaces, originally stated in [3, Exercise 6.8.10].

Theorem 4.11: Characterisation of Cauchy completeness

A metric space (E, d) is complete in the classical sense if and only if every distributor $\mathcal{I} \nrightarrow \mathcal{E}$ which has a right adjoint is induced by an $\bar{R}_+$-functor $\mathcal{I} \to \mathcal{E}$.

Proof:

(⇒) Suppose that (E, d) is Cauchy-complete. Then, using the proof of Theorem 4.10, we can construct a distributor $F_a : \mathcal{I} \nrightarrow \mathcal{E}$ for every a by taking

$$F_a(x) = d(a, x)$$

We know that each of these distributors has a right adjoint, and since the elements of a complete metric space E are in bijection with the distributors F_a for all $a \in E$, we know that there are no other distributors available. Now, recall that since $\mathcal{I}$ is the singleton metric space, there is a unique $\bar{R}_+$-functor $F_a^* : \mathcal{I} \to \mathcal{E}$ defined as follows:

$$F_a^*(*) = a$$

Indeed, the distributor F_a is induced by this functor by taking

$$F_a(B, A) = \mathcal{E}(B, F_a^*(A))$$

(⇐) Suppose that every distributor $F_a : \mathcal{I} \nrightarrow \mathcal{E}$ is induced by a $\bar{R}_+$-functor $F_a^* : \mathcal{I} \to \mathcal{E}$. Again, since $\mathcal{I}$ is the singleton metric space, we know that there are exactly $|E|$ such functors, defined using the formula

$$F_a^*(*) = a$$

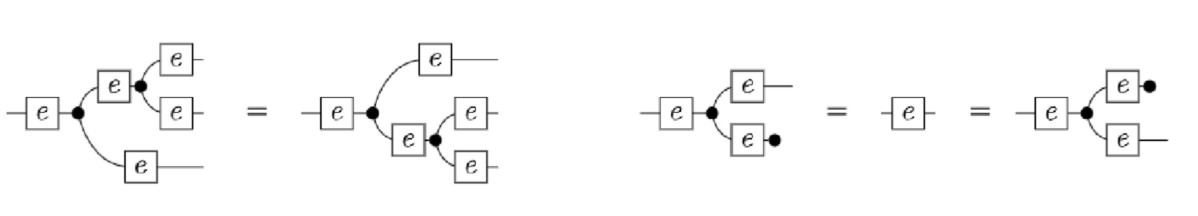
We can now observe based on [2, Section 2] that every induced distributor must have a right adjoint given by

$$G_a(A, B) = \mathcal{E}(F_a^*(A), B)$$

And thus by Theorem 4.10 we conclude that since we have constructed exactly $|E|$ distinct pairs of adjoint distributors, and these must be in bijection with the Cauchy-completion of the metric space (E, d), then also the metric space itself must be complete. □

Question 5

By doing a little research, find the definition of *Karoubi envelope* (also called *idempotent completion*) of a category. Suppose now that $\mathcal{C}$ is a copy-discard category. Suppose that every idempotent morphism e of $\mathcal{C}$ satisfies the following equations.



Find a canonical extension of the copy-discard structure of $\mathcal{C}$ to its Karoubi envelope.

In this section, we will interpret the phrase "canonical extension" to simply mean a straightforward extension that is obvious from the definition, as no technical meaning has been attributed to this term in the question. We will start our construction by defining the Karoubi envelope of a category [9]:

Definition 5.1: Karoubi envelope

Let $\mathcal{C}$ be a category. The Karoubi envelope of $\mathcal{C}$ is the category $\bar{\mathcal{C}}$ defined as follows:

- $\mathrm{Ob}(\bar{\mathcal{C}})$ are all pairs (X, e), where $X \in \mathcal{C}$ and $e \in \mathcal{C}(X, X)$ is idempotent
- Morphisms $\bar{\mathcal{C}}((X, e), (X', e'))$ are given by all morphisms $f \in \mathcal{C}(X, X')$ such that $f = e' \circ f \circ e$ (or equivalently $e' \circ f = f = f \circ e$)
- For any object (X, e), the identity morphism is e
- Composition is the same as composition in $\mathcal{C}$

It is worth noting that if $\mathcal{C}$ is a monoidal category, then its Karoubi envelope inherits the monoidal structure as well in the obvious way [4, Proposition 4.5.2]. Let us now turn our attention to the copy-discard category. Recall that a copy-discard category is defined as a monoidal category $(\mathcal{C}, \otimes, I)$ where for each object $X \in \mathcal{C}$ there exist two morphisms $\mathrm{copy}_X : X \to X \otimes X$ and $\mathrm{del}_X : X \to I$ satisfying the counitality and coassociativity laws as follows [10]:

$$(\mathrm{del}_X \otimes \mathrm{id}_X) \circ \mathrm{copy}_X = (\mathrm{id}_X \otimes \mathrm{del}_X) \circ \mathrm{copy}_X = \mathrm{id}_X \tag{6}$$

$$(\mathrm{id}_X \otimes \mathrm{copy}_X) \circ \mathrm{copy}_X = (\mathrm{copy}_X \otimes \mathrm{id}_X) \circ \mathrm{copy}_X \tag{7}$$

The equations given to us in the problem exactly correspond to the coassociativity and counitality laws, except that there is an extra morphism e on every wire. Specifically, the laws given tell us the following:

$$(\mathrm{del}_X \otimes \mathrm{id}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = (\mathrm{id}_X \otimes \mathrm{del}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = e \tag{8}$$

$$(e \otimes e \otimes \mathrm{id}_X) \circ (\mathrm{copy}_X \otimes \mathrm{id}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = (\mathrm{id}_X \otimes e \otimes e) \circ (\mathrm{id}_X \otimes \mathrm{copy}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e \tag{9}$$

We will now construct an extension of the copy-discard structure to the Karoubi envelope of $\mathcal{C}$. We would like to define terms $\mathrm{copy}_{(X,e)}$ and $\mathrm{del}_{(X,e)}$ for all $(X, e) \in \bar{\mathcal{C}}$ such that the following equations hold:

$$(\mathrm{del}_{(X,e)} \otimes \mathrm{id}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = (\mathrm{id}_{(X,e)} \otimes \mathrm{del}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = \mathrm{id}_{(X,e)} \tag{10}$$

$$(\mathrm{id}_{(X,e)} \otimes \mathrm{copy}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = (\mathrm{copy}_{(X,e)} \otimes \mathrm{id}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} \tag{11}$$

Note that since $\mathrm{id}_{(X,e)} = e$ by Definition 5.1, it suffices to satisfy the following:

$$(\mathrm{del}_{(X,e)} \otimes e) \circ \mathrm{copy}_{(X,e)} = (e \otimes \mathrm{del}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = e \tag{12}$$

$$(e \otimes \mathrm{copy}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = (\mathrm{copy}_{(X,e)} \otimes e) \circ \mathrm{copy}_{(X,e)} \tag{13}$$

We may now compare equation 8 with 12 as well as 9 with 13 to come up with an extension of the copy-discard structure to the Karoubi envelope.

Definition 5.2: Copy-discard structure on $\bar{\mathcal{C}}$

Let $\mathcal{C}$ be a category and $\bar{\mathcal{C}}$ be its Karoubi envelope. Define the copy-discard structure on $\bar{\mathcal{C}}$ as follows:

- $\mathrm{copy}_{(X,e)} = (e \otimes e) \circ \mathrm{copy}_X \circ e$
- $\mathrm{del}_{(X,e)} = \mathrm{del}_X \circ e$

Intuitively, this takes the standard copy operation and applies e before and after its application, while simply applying e before the normal delete operation. The following theorem will prove that this construction works as intended, and is the canonical extension of the copy-discard structure to the Karoubi envelope of a category.

Theorem 5.3: The copy-discard structure works

Let all of the terms be defined as in Definition 5.2. The copy-discard structure is well-defined and satisfies the properties given in equations 10 and 11.

Proof:

We must first ensure that the defined morphisms are actually morphisms in $\bar{\mathcal{C}}$. Recall that since $\text{copy}_{(X,e)} : (X,e) \to ((X,e) \otimes (X,e))$ and $\text{del}_{(X,e)} : (X,e) \to I$, they must satisfy the following conditions in order to be morphisms in $\bar{\mathcal{C}}$:

$$(e \otimes e) \circ \text{copy}_{(X,e)} \circ e = \text{copy}_X$$
$$\text{id}_I \circ \text{del}_{(X,e)} \circ e = \text{del}_X$$

This is easily seen as

$$(e \otimes e) \circ \text{copy}_{(X,e)} \circ e = (e \otimes e) \circ (e \otimes e) \circ \text{copy}_X \circ e \circ e = (e \otimes e) \circ \text{copy}_X \circ e = \text{copy}_{(X,e)}$$
$$\text{id}_I \circ \text{del}_{(X,e)} \circ e = \text{id}_I \circ \text{del}_X \circ e \circ e = \text{del}_X \circ e = \text{del}_{(X,e)}$$

Hence both morphisms are well-defined.

We will now move onto proving that the defined morphisms satisfy the properties that we want. Let us start with the counitality condition. We can see that

$$\begin{aligned}
&(\text{del}_{(X,e)} \otimes \text{id}_{(X,e)}) \circ \text{copy}_{(X,e)} = \\
&= ((\text{del}_X \circ e) \otimes e) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= ((\text{del}_X \circ e) \otimes (\text{id}_X \circ e)) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= (\text{del}_X \otimes \text{id}_X) \circ (e \otimes e) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= (\text{del}_X \otimes \text{id}_X) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= e = \quad \text{(above and below by equation 8)} \\
&= (\text{id}_X \otimes \text{del}_X) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= (\text{id}_X \otimes \text{del}_X) \circ (e \otimes e) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= ((\text{id}_X \circ e) \otimes (\text{del}_X \circ e)) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= (e \otimes (\text{del}_X \circ e)) \circ (e \otimes e) \circ \text{copy}_X \circ e = \\
&= (\text{id}_{(X,e)} \otimes \text{del}_{(X,e)}) \circ \text{copy}_{(X,e)}
\end{aligned}$$

Hence, the counitality condition also holds for the extension.

We can now similarly prove the coassociativity condition:

$$
\begin{aligned}
&(\mathrm{id}_{(X,e)} \otimes \mathrm{copy}_{(X,e)}) \circ \mathrm{copy}_{(X,e)} = \\
&= (e \otimes ((e \otimes e) \circ \mathrm{copy}_X \circ e)) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= ((\mathrm{id}_X \circ \mathrm{id}_X \circ e) \otimes ((e \otimes e) \circ \mathrm{copy}_X \circ e)) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (\mathrm{id}_X \otimes e \otimes e) \circ (\mathrm{id}_X \otimes \mathrm{copy}_X) \circ (e \otimes e) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (\mathrm{id}_X \otimes e \otimes e) \circ (\mathrm{id}_X \otimes \mathrm{copy}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \quad \text{(by equation 9)} \\
&= (e \otimes e \otimes \mathrm{id}_X) \circ (\mathrm{copy}_X \otimes \mathrm{id}_X) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (e \otimes e \otimes \mathrm{id}_X) \circ (\mathrm{copy}_X \otimes \mathrm{id}_X) \circ (e \otimes e) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (((e \otimes e) \circ \mathrm{copy}_X \circ e) \otimes (\mathrm{id}_X \circ \mathrm{id}_X \circ e)) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (((e \otimes e) \circ \mathrm{copy}_X \circ e) \otimes \circ e) \circ (e \otimes e) \circ \mathrm{copy}_X \circ e = \\
&= (\mathrm{copy}_{(X,e)} \otimes \mathrm{id}_{(X,e)}) \circ \mathrm{copy}_{(X,e)}
\end{aligned}
$$

Thus completing the proof that the copy-discard structure is valid and is a canonical extension of the structure from $\mathcal{C}$ to $\bar{\mathcal{C}}$. □

Conclusion

In this mini-project, we have investigated the concept of Cauchy completion and completeness across various different domains of category theory. We have seen that, although the concept is similar throughout, its interpretation differs depending on the specific type of category chosen. In the concrete example of matrices and stochastic matrices, Cauchy completeness can be interpreted as the relationship between the projection and embedding mappings in a vector space, which are the "splittings" of an idemponent matrix. On the other hand, in the case of metric spaces, which are encoded as enriched categories, Cauchy completeness can be interpreted directly (as a Cauchy-complete vector space in the normal sense) as well as in the category-theoretical sense of a splitting. Indeed, although not defined in this mini-project, it is possible to prove that adjoint distributors seen in Definition 4.4 satisfy a condition which is equivalent to the splitting of idempotents in the category of modules [2]. This is quite curious, as it shows that this is, in fact, the same construction in a different category.
We have also seen a nice characterisation of when an idempotent morphism splits using a representable presheaf, and concluded by investigating how to extend the categorical copy-discard structure of a category to a different category, namely its Karoubi envelope.

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