The University of Oxford

MSc (Computer Science / Mathematics and Foundations of Computer Science)

Applied Category Theory

Trinity Term 2025

*You have to choose exactly **one** of the following short projects.*

Project 1: Cauchy Completion

A morphism $e : X \to X$ in a category $\mathbf{C}$ is called *idempotent* if $e \circ e = e$. A *splitting* of an idempotent morphism $e : X \to X$ is an object E together with maps $\iota : E \to X$ and $\pi : X \to E$ such that $\pi \circ \iota = \mathrm{id}_E$ and $\iota \circ \pi = e$. A category is called *Cauchy-complete* if every idempotent has a splitting. Write a short project (10-12 pages) about this concept, which should address the following points.

1. Show that the category of (finite, real-entried) matrices is Cauchy-complete. Give a geometric interpretation of the splitting of idempotents.

2. Is the subcategory of stochastic matrices Cauchy-complete as well? If you can, give an interpretation in terms of Markov chains. (If you are not familiar with Markov chains, you can reason purely in terms of matrices, as in the previous point.)

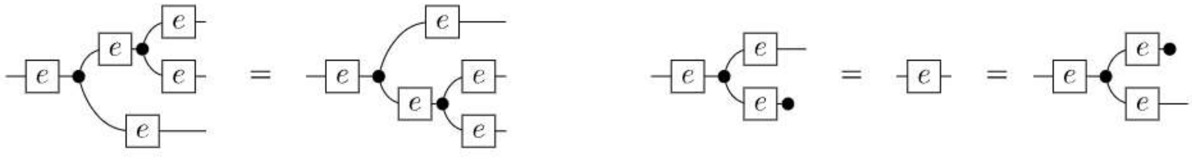
3. Given an idempotent morphism $e : X \to X$, define the *presheaf of left-invariant morphisms* $\mathrm{Inv}_e : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ as follows,

$$\mathrm{Inv}_e(Y) \;=\; \{p : Y \to X \mid e \circ p = p\}$$

   with its action on morphisms given by precomposition. Show that e splits if and only if Inv_e is representable. Give an interpretation in terms of "virtual objects".

4. By doing a little research, find out what a Cauchy-complete $\mathbf{V}$-category is, and explain how the notion relates to completeness of metric spaces.

5. (For maximum grade.) By doing a little research, find the definition of *Karoubi envelope* (also called *idempotent completion*) of a category. Suppose now that $\mathbf{C}$ is a copy-discard category. Suppose that every idempotent morphism e of $\mathcal{C}$ satisfies the following equations.



   Find a a canonical extension of the copy-discard structure of $\mathbf{C}$ to its Karoubi envelope. (Hint: it might be helpful to think of "virtual objects" once again.)

Question 2: The Continuation Monad

On the category of sets (and similar categories used in computer science), the *continuation monad with result object* R assigns to an object X the object $[[X, R], R]$, where $[-, -]$ denotes the (internal) hom-set. Write a short project (10-12 pages) about this monad (and about a closely related one, see below) which should address the following points.

1. By doing a little research, familiarize yourself with this monad, and write explicitly its structure (either in terms of unit and multiplication, or in terms of bind and return).
2. Generalize the construction of this monad to categories which do not admit an internal hom, but which have all (not just finite) products. (Your definition should coincide with the usual one when instantiated in the category of sets.)
3. On such a category with products, denote the monad defined in the previous point by C_R. Let T be any other monad, and suppose that the result object R has a T-algebra structure $r : TR \to R$. Form a natural transformation $T \Rightarrow C_R$ compatible with the monad structures. Give an example of this natural transformation in functional programming (if you are a computer scientist), or in probability/functional analysis (if you are a mathematician).
4. By doing a little research, explain the informal statement that *every monad is a submonad of a continuation.* Then, given a monad T on a category with products, give sufficient conditions on a T-algebra R which make the morphism $T \Rightarrow C_R$ monic. (Hint: keep in mind the example you gave in the previous point.)
5. (For maximum grade.) How does the monad C_R depend on the choice of the object R? Is it functorial in R? Is the unit of the monad natural in R? If you can, give an interpretation of this dependence on R in terms of polymorphism.

- Please typeset your project using LaTeX or a similar system. Diagrams and string diagrams in LaTeX can be created using interfaces such as *Quiver* (`https://q.uiver.app`) and *TikZit* (`https://tikzit.github.io/`).
- Make sure to cite any sources you use, including blogs or websites such as the nLab.
- Feel free to use any AI tools to help you, but use your critical thinking: hallucinations in category theory are still quite frequent. Check independently any mathematical assertions. Keep also in mind that the output of an AI tool is not an acceptable reference to cite.